COMPUTATIONAL MECHANICS
WCCM VI in conjunction with APCOM'04, *Sept. 5-10, 2004, Beijing, China*

Analysis of Stabilization Operators in a Galerkin Least-Squares Finite Element Discretization of the Incompressible Navier-Stokes Equations

M. Polner*, J.J.W. van der Vegt, R.M.J. van Dammer

Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands
e-mail: {m.polner, j.j.w.vandervegt, r.m.j.vandamme}@math.utwente.nl

Abstract In this paper the design and analysis of a dimensionally consistent stabilization operator for a time-discontinuous Galerkin least-squares finite element method for unsteady viscous flow problems governed by the incompressible Navier-Stokes equations, is discussed. The analysis results in a class of stabilization operators which satisfy essential conditions for the stability of the numerical discretization.

Key words: Galerkin least-squares finite element methods, stabilization operator, incompressible Navier-Stokes equations

INTRODUCTION

In this paper we address the issue of developing a numerical scheme which is suitable for a wide range of unsteady viscous flow problems governed by the incompressible Navier-Stokes equations. We focus on stabilized finite element methods since the concept of a stabilization operator eliminates for incompressible flows the complications of designing elements which satisfy the inf-sup stability condition. The motivation of the present study is the need to understand the origins of stabilized methods and stabilization operators. Stabilized methods for convection dominated flows were introduced by Hughes and Brooks [2]. In [3] the design of a stabilization parameter is confirmed to be a crucial ingredient for simulating the advective-diffusive model and some improved possibilities are suggested. The question of a careful design for the stability parameter is re-addressed and further improvements are tested in [4].

The starting point of our discussion are the symmetrized Navier-Stokes equations using entropy variables, as derived in [7]. The symmetrized equations using entropy variables in compressible flow, which have been investigated by Godunov [5], Harten [6], Hughes et al. [8], result in a global entropy stability which is automatically inherited by the numerical discretization, see for instance Shakib et al. [13] or Barth [1]. The concept of symmetrization is also important for incompressible flows since this provides a good starting point for the stabilized finite element formulation, which is one of the topics of this paper.

In this paper we address two issues. The first one is the design of the stabilization operator in the Galerkin least-squares finite element method, which is critical for the accuracy and stability of the discretization. In [7] a stabilization operator is proposed as a natural extension of previous research on incompressible flows using primitive variables [4]. The extensions suggested are, however, based on numerical experiments, without a detailed analysis of its properties. In this paper we will give therefore a consistent mathematical derivation of the stabilization operator. First, we will use primitive variables in the construction of the stabilization matrix and conduct a dimension analysis to determine its dependence on the flow variables. Next, the systematic derivation of the stabilization operator for the primitive variables will be extended to a more general set of variables, which also yields the stabilization operator suggested in [7].

The second topic of this paper is to give an outline of the analysis of the resulting stabilization operator such that we can ensure stability and coercivity of the Galerkin least-squares finite element discretization, at least for the locally linearized problem. This analysis is an extension of the work in [4] to the space-time formulation of the linearized incompressible Navier-Stokes equations in the symmetrized formulation derived in [7]. For more details we refer to [12]. This proof provides additional information on the admissible stabilization operators.

This paper is organized as follows: in Section 2 we discuss the symmetrized formulation of the incompressible Navier-Stokes equations. Section 3 describes the Galerkin least-squares finite element method for the symmetrized incompressible Navier-Stokes equations. Section 4 presents a dimensional analysis to obtain a suitable stabilization operator. Finally, in Section 5 we state conditions for the coercivity of the Galerkin least-squares finite element discretization for the linearized case. We conclude with a summary of the main results and some remarks.

THE GOVERNING EQUATIONS

Consider the incompressible Navier-Stokes equations in a time-dependent flow domain $\Omega(t)$. Since the flow domain boundary is moving and deforming in time, we do not make a separation between the space and time variables and consider directly the space $\mathbb{R}^{d+1}$, where d is the number of space dimensions. Assume that $d = 3$. Let $\mathcal{E} \subset \mathbb{R}^4$ be an open, bounded space-time domain. A point $x \in \mathbb{R}^4$ has coordinates (x_0, x_1, x_2, x_3), with $x_0 = t$ representing time. The flow domain $\Omega(t) \subset \mathcal{E}$ at time t is defined as: $\Omega(t) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (t, x_1, x_2, x_3) \in \mathcal{E}\}$. The space-time domain boundary $\partial\mathcal{E}$ consists of the hypersurfaces $\Omega(t_0) = \{x \in \partial\mathcal{E} \mid x_0 = 0\}$, $\Omega(T) = \{x \in \partial\mathcal{E} \mid x_0 = T\}$, and $\mathcal{Q} = \{x \in \partial\mathcal{E} \mid 0 < x_0 < T\}$.

Let $Y : \mathcal{E} \mapsto \mathbb{R}^5$ denote the vector of primitive variables $(p, u_1, u_2, u_3, T)^T$ and $F : \mathbb{R}^5 \mapsto \mathbb{R}^{5\times4}$ denotes the flux tensor, with the flux vector in the ℓth coordinate direction F_ℓ ($\ell = 0, \dots, 3$) given by the columns of F, i.e.,

$$F = \begin{pmatrix} 0 & u_1 & u_2 & u_3 \\ u_1 & u_1^2 + p & u_1u_2 & u_1u_3 \\ u_2 & u_1u_2 & u_2^2 + p & u_2u_3 \\ u_3 & u_1u_3 & u_2u_3 & u_3^2 + p \\ T & u_1T & u_2T & u_3T \end{pmatrix} \tag{1}$$

where u_i denotes the velocity component in the ith Cartesian coordinate direction, p the pressure and T the temperature. Using these notations, the incompressible Navier-Stokes equations can be written in a conservative form as

$$F_\ell(Y(x))_{,\ell} + (K_{ij}(Y)Y_{,j})_{,i} = 0, \quad x \in \mathcal{E}, \tag{2}$$

where $K_{ij} \in \mathbb{R}^{5\times5}$ for $i, j = 1, 2, 3$ denote the viscous flux Jacobian matrices and the summation convention is used on repeated indices.

In [7], it is demonstrated that with the proper choice of variables, it is possible to obtain a symmetric form of the Navier-Stokes equations which is valid for both compressible and incompressible flows. As a starting point, we consider the symmetrized form of the Navier-Stokes equations in the incompressible limit, which are given as:

$$\tilde{A}_0 V_{,t} + \tilde{A}_{i,i} V_{,i} = (\tilde{K}_{ij} V_{,j})_{,i} \tag{3}$$

where the summation convention on repeated indices is used. The set of variables V are the so-called entropy variables given as

$$V = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{pmatrix} = \frac{1}{T} \begin{pmatrix} \tilde{\mu} - |u|^2/2 \\ u_1 \\ u_2 \\ u_3 \\ -1 \end{pmatrix}, \tag{4}$$

where $\tilde{\mu}$ is the chemical potential, defined as $\tilde{\mu} = e + p/\rho - Ts$, e denotes the internal energy, ρ the density and s the specific entropy. The most important feature of the symmetrizing variables is that the flux Jacobian matrices in the incompressible limit have the following properties: $\tilde{A}_0$ is symmetric positive-semidefinite, $\tilde{A}_i$ is symmetric, $K = [\tilde{K}_{ij}]$ is symmetric (i.e., $\tilde{K}_{ij} = \tilde{K}_{ji}^T$) and positive-semidefinite for all $i, j = 1, 2, 3$. The $\tilde{A}_i$ matrices are given in the Appendix since they are needed for the analysis. It can be shown (see [7]) that the symmetrized Navier-Stokes equations obtained in the incompressible limit are identical to the incompressible Navier-Stokes equations and the temperature equation, but provide a much better starting point for the finite element discretization.

GALERKIN LEAST-SQUARES FINITE ELEMENT FORMULATION

Consider a partition of the time interval $I = (0, T)$ using the time levels $0 = t_0 < t_1 < \ldots < t_N = T$ and we denote by $I_n = (t_n, t_{n+1})$ the nth time interval. A space-time slab is defined as $\mathcal{E}_n = \mathcal{E} \cap I_n$. In each space-time slab $\mathcal{E}_n$ we define a partition $\mathcal{T}_h^n$ into $(n_{el})_n$ non-overlapping elements $\mathcal{E}_n^e$. The space-time elements $\mathcal{E}_n^e$ are obtained by splitting the spatial domain $\Omega(t_n)$ into a set of non-overlapping elements Ω_n^e and connecting them with a mapping Φ_t^n to the elements $\Omega_{n+1}^e \subset \Omega(t_{n+1})$ at time t_{n+1}. Within each space-time element the trial and test functions are approximated by kth-order polynomials $\mathcal{P}_k$. The trial function space is denoted by V_h and the test function space by W_h. Their elements are assumed to be $\mathcal{C}^0$ continuous within each space-time slab, but discontinuous across the interfaces of the space-time slabs, namely at times $t_1, t_2, \ldots, t_{N-1}$.

Let us recall the Galerkin least-squares variational formulation for the Navier-Stokes equations in terms of entropy variables. Within each space-time slab $\mathcal{E}_n$, find a $V \in V_h$ such that for all $W \in W_h$ the following relation is satisfied

$$\int_{\mathcal{E}_n} W \cdot \left(F_\ell(V)_{,\ell} - (\tilde{K}_{ij} V_{,j})_{,i} \right) \mathrm{d}\mathcal{E} + B_{ls}(V, W) = 0, \quad \ell = 0, \ldots, 3, \tag{5}$$

where the second term is the least-squares stabilization operator defined as

$$B_{ls}(V, W) = \sum_{e=1}^{(n_{el})_n} \int_{\mathcal{E}_n^e} (\mathcal{L}_V V) \cdot \tilde{\tau} (\mathcal{L}_V W) \mathrm{d}\mathcal{E}, \tag{6}$$

with $\mathcal{L}_V$ the symmetrized Navier-Stokes operator:

$$\mathcal{L}_V = \tilde{A}_\ell \frac{\partial}{\partial x_\ell} - \frac{\partial}{\partial x_i}(\tilde{K}_{ij} \frac{\partial}{\partial x_j}) \quad \text{for } \ell = 0, \ldots, 3, \ i, j = 1, 2, 3. \tag{7}$$

We use also the notation $\mathcal{L}_V^{inv}$ to denote the inviscid counterpart of $\mathcal{L}_V$, (hence $\tilde{K}_{ij} = 0$ for all $i, j = 1, 2, 3$.) The stabilization operator is added in order to prevent numerical oscillations in regions with strong gradients which are not well represented on the computational mesh. In the least-squares operator, the choice for the $\tilde{\tau}$ matrix is crucial, and is examined in detail in this paper. This operator greatly influences the stability of the numerical scheme. Note that the least-squares integral is only defined on the interior of the elements.

EXPLICIT CONSTRUCTION OF STABILIZATION OPERATORS

In this section a class of dimensionally consistent stabilization operators will be derived, which also includes, as a special case, the stabilization operator given in [7]. We recall that the standard definition of the stabilization matrix requires $\tilde{\tau}$ to be symmetric, positive definite, have dimensions of time, and scale linearly with the element size (see [9]). Due to the fully coupled structure of the system, it is very difficult to define a stabilization matrix directly in terms of the entropy variables. Therefore, the choice of variables Y in which the system is expressed and used to define the stabilization operator is important.

Our starting point is a dimensional analysis of the stabilization matrix τ_Y related to the primitive variables $Y = (p, u_1, u_2, u_3, T)^T$. This stabilization operator is related to $\tilde{\tau}$ through the transformation

$$\tilde{\tau} = V_{,Y} \tau_Y. \tag{8}$$

For brevity of our analysis, we consider only the Euler equations, however the analysis can be extended straightforwardly to the Navier-Stokes equations. First we introduce some notation. Consider the set S of all flow variables (such as velocity, temperature, pressure, etc.), and its power set, denoted by $P(S)$.

Definition 1. *Given the set S, $P(S)$, a set $\mathcal{V} = \{\nu_1, \nu_2, \ldots, \nu_n\} \in P(S)$ and a set of functionals $F = \{f : P(S) \to \mathbb{R}\}$ such that addition on F is defined only among the elements which have the same physical dimension. Furthermore, define the following mapping*

$$\Lambda : (\nu_1, \nu_2, \ldots, \nu_n) \mapsto (\lambda_1\nu_1, \lambda_2\nu_2, \ldots, \lambda_n\nu_n) \text{ for } \lambda_i > 0, \tag{9}$$

such that

$$f(\Lambda(\mathcal{V})) = \lambda_1^{m_1} \lambda_2^{m_2} \cdots \lambda_n^{m_n} f(\mathcal{V}), \quad \forall f \in F. \tag{10}$$

Then, an equivalence relation $\sim_\mathcal{V}$ over the set of functionals F is defined as:

$$f \sim_\mathcal{V} g \quad \Longleftrightarrow \quad \text{whenever} \begin{cases} f(\Lambda(\mathcal{V})) & = \lambda_1^{m_1} \lambda_2^{m_2} \cdots \lambda_n^{m_n} f(\mathcal{V}) \\ g(\Lambda(\mathcal{V})) & = \lambda_1^{k_1} \lambda_2^{k_2} \cdots \lambda_n^{k_n} g(\mathcal{V}), \end{cases} \tag{11}$$

$$\text{it follows that } m_i = k_i, \ \forall i = 1, \ldots n. \tag{12}$$

We say that f is dimensionally equivalent (or has the same dimension) to g with respect to the set of flow variables $\mathcal{V}$. An equivalence class is a subset of F of the form $\{g : g \sim_\mathcal{V} f\}$, where f is an element in F. This equivalence class we denote by

$$[f]_\mathcal{V} = |\nu_1|^{m_1} |\nu_2|^{m_2} \cdots |\nu_n|^{m_n}. \tag{13}$$

Definition 2. *Let $A = (a_{ij}) \in \mathbb{R}^{n\times m}$, $B = (b_{ij}) \in \mathbb{R}^{n\times m}$. Then, we define $A \sim_{\mathcal{V}} B$ if*

$$a_{ij} \sim_{\mathcal{V}} b_{ij} \quad \textit{for all} \quad i = 1,\dots,n,\ j = 1,\dots,m. \tag{14}$$

Using these definitions, we determine the dimension of the Euler equations formulated in primitive variables. We first transform the equations $\mathcal{L}_V^{inv}(V) = 0$ to the primitive variables $Y = (p, u_1, u_2, u_3, T)^T$:

$$A_0(Y)Y_{,t} + A_i(Y)Y_{,i} = 0. \tag{15}$$

The coefficient matrices $A_0(Y), A_i(Y)$ for the incompressible Navier-Stokes equations in primitive variables are given in the Appendix. Next, we give a velocity dimension which the stabilization matrix must satisfy. Introduce $U = |u|$ and let $\mathcal{V} = \{u\}$ in Definition 1. Then, for each component of the velocity $[u_i]_u = U$, $i = 1, 2, 3$. According to the momentum equations $[p_{,i}]_u = U$ and from the equation for the temperature field it follows that $[T_{,i}]_u = 1/U$. Hence, the following dimensional equivalence is valid

$$[Y_{,i}]_u = \begin{pmatrix} U \\ 1 \\ 1 \\ 1 \\ 1/U \end{pmatrix},\ [A_0(Y)]_u = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & U & U & U & U^2 \end{pmatrix},\ [A_i(Y)]_u = \begin{pmatrix} 0 & \delta_{1i} & \delta_{2i} & \delta_{3i} & 0 \\ \delta_{1i} & U & \delta_{2i}U & \delta_{3i}U & 0 \\ \delta_{2i} & \delta_{1i}U & U & \delta_{3i}U & 0 \\ \delta_{3i} & \delta_{1i}U & \delta_{2i}U & U & 0 \\ U & U^2 & U^2 & U^2 & U^3 \end{pmatrix}$$

for $i = 1, 2, 3$ and with δ_{ij} the Kronecker delta symbol. Consequently, for the system of equations we have the dimension:

$$[A_0(Y)Y_{,t} + A_i(Y)Y_{,i}]_u = \begin{pmatrix} 1 \\ U \\ U \\ U \\ U^2 \end{pmatrix}. \tag{16}$$

Having established in (5) a Galerkin least-squares method for entropy variables, we may transform it to primitive variables. All terms in the variational formulation (5) remain essentially unchanged with the least-squares contribution written in terms of the differential operator

$$\mathcal{L}_Y{}^{inv} = A_0(Y)\frac{\partial}{\partial t} + A_i(Y)\frac{\partial}{\partial x_i}, \quad i = 1, 2, 3. \tag{17}$$

Our aim is now to construct a stabilized finite element method, which admits discrete solutions Y^h with the same dimension as the solution Y of (15). In addition, we require that the stabilization operator must be dimensionally equivalent with the Galerkin and the boundary operator. Note here that similar assumptions are made for the discrete solutions in [14], where a scaling analysis is performed to determine the appropriate low Mach number behavior of the stabilization matrix. Therefore, these requirements provide an additional condition on the components of the stabilization matrix τ_Y, i.e.,

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & U & U & U & 0 \\ 1 & U & U & U & 0 \\ 1 & U & U & U & 0 \\ U & U^2 & U^2 & U^2 & U^3 \end{pmatrix} \cdot [\tau_Y]_u \begin{pmatrix} 1 \\ U \\ U \\ U \\ U^2 \end{pmatrix} = \begin{pmatrix} 1 \\ U \\ U \\ U \\ U^2 \end{pmatrix}. \tag{18}$$

From (18) it follows that a suitable stabilization τ_Y is dimensionally equivalent to the inverse of the matrix in (18), i.e.,

$$[\tau_Y]_u = \begin{pmatrix} U & 1 & 1 & 1 & 0 \\ 1 & 1/U & 1/U & 1/U & 0 \\ 1 & 1/U & 1/U & 1/U & 0 \\ 1 & 1/U & 1/U & 1/U & 0 \\ 1/U & 1/U^2 & 1/U^2 & 1/U^2 & 1/U^3 \end{pmatrix}. \tag{19}$$

In this way we have found how the stabilization matrix τ_Y depends on the magnitude of the velocity. In an analogous way we can obtain the density and temperature dimension of the stabilization matrix. Summarizing, the general form of the stabilization matrix in primitive variables is

$$[\tau_Y]_{\{\rho,u,T\}} = \begin{pmatrix} c_{11}U & c_{12} & c_{13} & c_{14} & 0 \\ \frac{c_{21}}{\rho} & \frac{c_{22}}{\rho U} & \frac{c_{23}}{\rho U} & \frac{c_{24}}{\rho U} & 0 \\ \frac{c_{31}}{\rho} & \frac{c_{32}}{\rho U} & \frac{c_{33}}{\rho U} & \frac{c_{34}}{\rho U} & 0 \\ \frac{c_{41}}{\rho} & \frac{c_{42}}{\rho U} & \frac{c_{43}}{\rho U} & \frac{c_{44}}{\rho U} & 0 \\ \frac{c_{51}}{\rho U} & \frac{c_{52}}{\rho U^2} & \frac{c_{53}}{\rho U^2} & \frac{c_{54}}{\rho U^2} & \frac{c_{55}}{\rho U^3} \end{pmatrix} \tag{20}$$

where U is the velocity dimension and c_{ij} are dimensionless quantities. In this form of the matrix we still have 21 unknowns which need to be specified.

In the remaining part of this section we will give a definition for the stabilization matrix τ_Y using (20) and the properties of symmetry and positive definiteness of $\tilde{\tau}$. We give only a sketch of the proof of the following theorem, for details see [12].

Theorem 1. *Given the stabilization matrix τ_Y for primitive variables in the dimensionally consistent form (20) which is related to a symmetric, positive definite stabilization matrix $\tilde{\tau}$ for the entropy variables through the transformation (8). Then, a unique class of suitable stabilization matrices τ_Y and $\tilde{\tau}$, which are also Galileian invariant, can be obtained explicitly.*

Proof. Consider the middle 3×3 block in (20), corresponding to the three momentum equations. The symmetry of $\tilde{\tau}$ implies that this block is symmetric. Moreover, using rotation invariance it leads to the fact that this block is a constant times the identity matrix, i.e., $c_{22} = c_{33} = c_{44} = c$ and $c_{23} = c_{32} = c_{24} = c_{42} = c_{34} = c_{43} = 0$. For simplicity we introduce the following notation for the diagonal entries in τ_Y,

$$\tau_c := c_{11}U, \quad \tau_m := \frac{c}{\rho U}, \quad \tau_e := \frac{c_{55}}{\rho U^3}. \tag{21}$$

Based on the assumptions stated in the theorem, we are able to obtain the relations

$$c_{12} = \rho u_1 \tau_m + c_{21}, \; c_{13} = \rho u_2 \tau_m + c_{31}, \; c_{14} = \rho u_3 \tau_m + c_{41}. \tag{22}$$

Since $\tau_m \neq 0$, it follows from the relations in (22) that there are at least three additional non-vanishing entries in the matrix τ_Y. Using (22) and again the rotation invariance property of the stabilization matrices, we obtain the following general form

$$\tau_Y = \begin{pmatrix} \tau_c & \alpha\rho u_1\tau_m & \alpha\rho u_2\tau_m & \alpha\rho u_3\tau_m & 0 \\ -(1-\alpha)u_1\tau_m & \tau_m & 0 & 0 & 0 \\ -(1-\alpha)u_2\tau_m & 0 & \tau_m & 0 & 0 \\ -(1-\alpha)u_3\tau_m & 0 & 0 & \tau_m & 0 \\ -(h-k)\tau_e & -u_1\tau_e & -u_2\tau_e & -u_3\tau_e & \tau_e \end{pmatrix}. \tag{23}$$

where $\alpha \in \mathbb{R}$ is a parameter, h the specific enthalpy and $k = |u|^2/2$.

In this way we have obtained a class of stabilization matrices (23) for the primitive variables. Using (8) with (23) it is straightforward to obtain a class of the stabilization matrices for the entropy variables, given explicitly in [12]. □

When setting $\alpha = 1$ in Theorem 1, we obtain the stabilization matrix propossed in [7] based on numerical experiments.

COERCIVITY ESTIMATE FOR LINEARIZED NAVIER-STOKES EQUATIONS

The consistent derivation of the class of stabilization operators motivates the choice of the diagonal entries of (23) given in [7]. The following definition is a sufficient condition to prove the coercivity of the Galerkin least-squares finite element method for the entropy variables (see [12]), which is an essential requirement to ensure stability of the numerical discretization.

Definition 3. *The stabilization parameters τ_c, τ_m and τ_e on the element $\mathcal{E}_n^e$ are defined as*

$$\tau_c(x) = \frac{h_e|u(x)|_p}{2}, \ \tau_m(x) = \frac{h_e}{2\rho|u(x)|_p}\xi(Re_e(x)), \ \tau_e(x) = \frac{h_e}{2\rho c_v|u(x)|_p}\xi(Re_e(x))$$

for all $x \in \mathcal{E}_n^e$, with

$$Re_e(x) = \frac{m_k\rho|u(x)|_p h_e}{\mu(x)}, \quad m_k = \min\{1, C_k\}, \tag{24}$$

$$\xi(Re_e(x)) = \begin{cases} Re_e(x), & 0 \leq Re_e(x) < 1 \\ 1, & Re_e(x) \geq 1, \end{cases} \tag{25}$$

where μ is the fluid viscosity, h_e denotes the element diameter and C_k is a positive constant independent of physical properties and element diameter.

Let us reformulate the system (3) to a form which is more convenient for our stability analysis. If we define the spatial gradient operator as $\nabla = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})^T$, the system (3) in the domain $\mathcal{E}$ can be written in the following dimensionless form

$$\rho T(\hat{V}_{i,i} + u_i V_{5,i}) = 0, \quad \text{for } i = 1,2,3 \tag{26}$$

$$\hat{A}_\ell \hat{V}_{,\ell} - \frac{1}{Re}(\hat{K}_{ij}\hat{V}_{,j})_{,i} = -\rho T \begin{pmatrix} \nabla V_1 \\ u \cdot \nabla V_1 \end{pmatrix}, \quad \text{for } i,j = 1,2,3, \quad \ell = 0, \dots 3, \tag{27}$$

where the variable $\hat{V}$ is the image of V under the projection $\pi : \mathcal{E}^5 \to \mathcal{E}^4$ such that $\pi(V) = \hat{V} = (V_2, V_3, V_4, V_5)^T$, $\hat{A}_i$ and $\hat{K}_{ij}$ denote the lower right 4×4 part of $\tilde{A}_i$, $\tilde{K}_{ij} \in \mathbb{R}^{5\times 5}$, respectively, and Re denotes the Reynolds number. Note that (26) is identical to $u_{i,i} = 0$. For notational simplicity, we denote the test function by $W = (W_1, W_2, W_3, W_4, W_5)^T$ and decompose it in an analogous way as the entropy variables V. Hence, we denote by $\hat{W} = (W_2, W_3, W_4, W_5)^T$.

Definition 4. *For $\ell = 0, \dots, 3$, $i, j = 1,2,3$, define the following differential operators:*

$\hat{\mathcal{L}} : \mathcal{E}^4 \to \mathbb{R}^4$ *such that*

$$\hat{\mathcal{L}} = \hat{A}_\ell \frac{\partial}{\partial x_\ell} - \frac{1}{Re}\frac{\partial}{\partial x_i}\left(\hat{K}_{ij}\frac{\partial}{\partial x_j}\right), \tag{28}$$

and $\hat{\mathcal{L}}^{inv}$ denotes the inviscid part of $\hat{\mathcal{L}}$. Futhermore, let $\hat{\mathcal{D}} : \mathcal{E}^4 \to \mathbb{R}$ such that

$$\hat{\mathcal{D}}\hat{V} = \rho T\left(\frac{\partial \hat{V}_i}{\partial x_i} + u_i \frac{\partial \hat{V}_4}{\partial x_i}\right), \tag{29}$$

and $\hat{\mathcal{F}} : \mathcal{E} \to \mathbb{R}^4$ *such that*

$$\mathcal{F}V_1 = -\rho T \begin{pmatrix} \nabla V_1 \\ u \cdot \nabla V_1 \end{pmatrix}. \tag{30}$$

The splitting of the original system (3) into (26) and (27) suggests the splitting of the stabilization matrix $\tilde{\tau}$, in the following way

$$\tilde{\tau} = \begin{pmatrix} \delta & \sigma \hat{V}^T \\ \sigma \hat{V} & \hat{\tau} \end{pmatrix} \tag{31}$$

where $\tilde{\tau}_{11} = \delta = \delta_1 \|\hat{V}\|^2$, $\sigma = c_\sigma \tau_e$, with the coefficient c_σ depending on the set of flow variables $\mathcal{V}$, i.e., $c_\sigma = c_\sigma(\mathcal{V})$ for some $\mathcal{V} \in P(S)$, and $\hat{\tau}$ is the lower right 4×4 submatrix of $\tilde{\tau}$. Note that this spliting is valid for all $\tilde{\tau}$ in the class of stabilization matrices obtained in the previous section, with different δ and σ. We can therefore consider $\tilde{\tau}$ in (31) being the representative of this class. For all cases $\delta > 0$, and $\hat{\tau}$ is symmetric positive definite. From Definition 3 it follows that $\tau_m = c_v \tau_e$. Hence, we can write the smallest and the largest eigenvalues of $\hat{\tau}$ as: $\lambda_{\min} = c_{\min} \tau_m = c_{\min} c_v \tau_e$ and $\lambda_{max} = c_{\max} \tau_e = \frac{c_{\max}}{c_v} \tau_m$, where $c_{\min}$ and $c_{\max}$ are positive and functions of $|u|, T$ and c_v.

We now introduce some notation. With $(\cdot,\cdot)_\mathcal{D}$ we denote the L^2 inner product in the open domain $\mathcal{D} \subset \mathbb{R}$. In case of vector arguments, the L^2-inner product is defined as

$$\begin{aligned} (\cdot,\cdot)_\mathcal{D} &: \mathcal{D}^m \times \mathcal{D}^m \longrightarrow \mathbb{R} \\ (V, W)_\mathcal{D} &= \int_\mathcal{D} W^T V \, \mathrm{d}\,\mathcal{E}, \quad \text{for all } V, W \in \mathcal{D}^m \end{aligned}$$

and $\|\cdot\|_{0,\mathcal{D}}$ is the corresponding norm in the space $L^2(\mathcal{D})$.

Let us reformulate the variational formulation (5) for the system written in the form (26-27). Within each space-time slab $\mathcal{E}_n$, find $(V_1, \hat{V}) \in V_h$ such that for all $(W_1, \hat{W}) \in W_h$ the following relation is satisfied

$$\begin{aligned} B(V,W) = {} & (\hat{A}_\ell \hat{V}_{,\ell} - \frac{1}{Re}(\hat{K}_{ij}\hat{V}_{,j})_{,i}, \hat{W})_{\mathcal{E}_n} + (\rho T(\hat{V}_{i,i} + u_i V_{5,i}), W_1)_{\mathcal{E}_n} + (-\mathcal{F}V_1, \hat{W})_{\mathcal{E}_n} + \\ & + \sum_e \Big(\hat{A}_\ell \hat{V}_{,\ell} + \rho T \begin{pmatrix} \nabla V_1 \\ u \cdot \nabla V_1 \end{pmatrix} - \frac{1}{Re}(\hat{K}_{ij}\hat{V}_{,j})_{,i}, \\ & \qquad \hat{\tau}(\hat{A}_\ell \hat{W}_{,\ell} + \rho T \begin{pmatrix} \nabla W_1 \\ u \cdot \nabla W_1 \end{pmatrix} - \frac{1}{Re}(\hat{K}_{ij}\hat{W}_{,j})_{,i}) \Big)_{\mathcal{E}_n^e} + \\ & + \sum_e \Big((\hat{\mathcal{D}}\hat{V}, \delta \hat{\mathcal{D}}\hat{W})_{\mathcal{E}_n^e} + \sigma \hat{\mathcal{D}}\hat{W}(\hat{V}, \hat{\mathcal{L}}\hat{V} - \mathcal{F}V_1)_{\mathcal{E}_n^e} + \sigma \hat{\mathcal{D}}\hat{V}(\hat{\mathcal{L}}\hat{W} - \mathcal{F}W_1, \hat{V})_{\mathcal{E}_n^e} \Big) = 0 \end{aligned}$$

If we assume that the Jacobian matrices $\hat{A}_\ell$ and $\hat{K}_{ij}$ are constant for all $\ell = 0, \ldots, 3$ and $i, j = 1, 2, 3$, then we can state the following coercivity result. For a proof we refer to [12].

Theorem 2. *There exist* $\epsilon_1 > 1$ *and* $\epsilon_2 > 0$ *such that for all* $(V_1, \hat{V}) \in V_h$

$$B(\hat{V}, V_1; \hat{V}, V_1) \geq \|V\|^2_{coers} \tag{32}$$

where the norm $\|\cdot\|^2_{coers}$ *is defined as*

$$\|V\|^2_{coers} = \frac{1}{Re}(\hat{K}_{ij}\hat{V}_{,j},\hat{V}_{,i})_{\mathcal{E}_n} - \left(\left(\frac{2\epsilon_1}{Re} - \frac{1}{Re^2}\right)c_{\max} + |c_\sigma|\frac{\epsilon_2}{Re}\right)\frac{1}{2\mu c_v}\|(\hat{K}_{ij}\hat{V})_{,i}\|^2_{0,\mathcal{E}_n} +$$

$$+\sum_e\left(\frac{\delta_1}{|\sigma|} - 2\epsilon_1\frac{|c_\sigma|}{c_{\min}c_v} - \frac{1}{\epsilon_2}\right)\|\sigma^{1/2}(\hat{\mathcal{D}}\hat{V})\|^2_{0,\mathcal{E}_n^e}\|\hat{V}\|^2_{0,\mathcal{E}_n} \quad (33)$$

$$+\sum_e(1-\frac{1}{\epsilon_1})\|\hat{\tau}^{1/2}(\mathcal{L}^{inv}\hat{V} - \mathcal{F}V_1)\|^2_{0,\mathcal{E}_n^e} + B_{bd}(\hat{V};V_1) \quad (34)$$

and $B_{bd}(\hat{V};V_1)$ *denotes the resulting natural boundary conditions.*

CONCLUDING REMARKS

In this article we have derived a class of stabilization operators for the incompressible Navier-Stokes equations using dimensional analysis. For this choice of stabilization operators we can give a coercivity estimate, at least for the locally linearized case, which is crucial for the stability of the finite element discretization.

Acknowledgement The nancial support from MARIN for the research of M. Polner is gratefully acknowledged.

APPENDIX

The advective flux Jacobians in terms of the entropy variables V in the incompressible limit have the form

$$\tilde{A}_0 = \rho T\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & u_1 \\ 0 & 0 & 1 & 0 & u_2 \\ 0 & 0 & 0 & 1 & u_3 \\ 0 & u_1 & u_2 & u_3 & r \end{pmatrix}, \quad \tilde{A}_1 = \rho T\begin{pmatrix} 0 & 1 & 0 & 0 & u_1 \\ 1 & 3u_1 & u_2 & u_3 & 2u_1^2+e_1 \\ 0 & u_2 & u_1 & 0 & 2u_1u_2 \\ 0 & u_3 & 0 & u_1 & 2u_1u_3 \\ u_1 & 2u_1^2+e_1 & 2u_1u_2 & 2u_1u_3 & u_1(r+2e_1) \end{pmatrix}$$

with $k = |u|^2/2$, $r = 2k + c_pT$ and $e_1 = h + k$. By symmetry, we can define $\tilde{A}_2$ and $\tilde{A}_3$, see [12]. The viscous flux Jacobians in terms of the entropy variables, which satisfy the relation $\tilde{K}_{ij} = \tilde{K}^T_{ji}$, are given in [7].
The Euler Jacobians with respect to the primitive variables $Y = (p, u_1, u_2, u_3, T)^T$ are:

$$\bar{A}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & \rho & 0 \\ 0 & \rho u_1 & \rho u_2 & \rho u_3 & \rho c_p \end{pmatrix}, \quad \bar{A}_1 = \begin{pmatrix} 0 & \rho & 0 & 0 & 0 \\ 1 & 2\rho u_1 & 0 & 0 & 0 \\ 0 & \rho u_2 & \rho u_1 & 0 & 0 \\ 0 & \rho u_3 & 0 & \rho u_1 & 0 \\ u_1 & \rho e_1 + \rho u_1^2 & \rho u_1 u_2 & \rho u_1 u_3 & \rho u_1 c_p \end{pmatrix},$$

Similarly, we can give $\bar{A}_2$ and $\bar{A}_3$, see [12]. The diffusivity coefficient matrices $\bar{K}_{ij}$, for $i, j = 1, 2, 3$ can be found for example in [7].

REFERENCES

[1] T.J. Barth, *Numerical methods for gasdynamic systems on unstructured meshes* , Notes in Computational Sci. and Eng., 5, 1998.

[2] A.N. Brooks, T.J.R Hughes, *Streamline upwind/Petrov-Galerkin methods for convective dominated flows with particular emphasis on the incompressible Navier-Stokes equations*, Comput. Meth. Appl. Mech. Engrg., 32, (1982), 199-259.

[3] L.P. Franca, S.L. Frey, T.J.R. Hughes, *Stabilized finite element methods: I. Application to the advective-diffusive model*, Comput. Meth. Appl. Mech. Engrg., 95, (1992), 253-276.

[4] L.P. Franca, S.L. Frey, *Stabilized finite element methods: II. The incompressible Navier-Stokes equations*, Comput. Meth. Appl. Mech. Engrg., 99, (1992), 209-233.

[5] S.K. Godunov, *The problem of generalized solution in the theory of quasilinear equations and in gas dynamics*, Russ. Math. Surveys, 17, (1962), 145-156.

[6] A. Harten, *On the symmetric form of systems of conservation laws with entropy*, J. Comp. Phys., 49, (1983), 151-164.

[7] G. Hauke, T.J.R. Hughes, *A comparative study of different sets of variables for solving compressible and incompressible flows*, Comput. Meth. Appl. Mech. Engrg, 153, (1998) 1-44.

[8] T.J.R. Hughes, L.P. Franca, M. Mallet, *A new finite element formulation for computational fluid dynamics: I. Symmetric forms of the compressible Euler and Navier-Stokes equations and the second law of thermodynamics*, Comput. Meth. Appl. Mech. Engrg, 54, (1986), 223-234.

[9] T.J.R. Hughes, M. Mallet, *A new finite element formulation for computational fluid dynamics: III. The generalized streamline operator for multidimensional advective-diffusive systems*, Comput. Meth. Appl. Mech. Engrg, 58, (1986), 305-328.

[10] A. Masud, T.J.R. Hughes, *A space-time Galerkin/least-squares finite element formulation of the Navier-Stokes equations for moving domain problems*, Comput. Meth. Appl. Mech. and Eng., 146(1-2) , (1997), 91-126.

[11] R.L. Panton, *Incompressible flow*, Wiley-Interscience, 1984.

[12] M. Polner, J.J.W. van der Vegt, R.M.J. van Damme, *Analysis of stabilization operators in a Galerkin least-squares finite element discretization of the incompressible Navier-Stokes equations*, in preparation.

[13] F. Shakib, T.J.R. Hughes, Z. Johan, *A new finite element formulation for computational fluid dynamics: X. The compressible Euler and Navier-Stokes equations*, Comput. Meth. Appl. Mech. and Eng., 89, (1991), 141-219.

[14] D.L. Wong, J.S. Darmofal, J. Peraire, *The solution of the compressible Euler equations at low Mach numbers using a stabilized finite element algorithm*, Comput. Meth. Appl. Mech. Engrg, 190, (2001), 5719-5737.